

Eigensolutions Sensitivity for Nonsymmetric Matrices with Repeated Eigenvalues

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A perturbation algorithm is developed for evaluating eigenvalue and eigenvector directional derivatives of nonsymmetric defective matrices. Some properties of these matrices are recalled; in particular, chains of generalized right and left eigenvectors and their orthogonal properties are defined. Small perturbations of the matrix are then considered. An asymptotic expansion of the eigensolutions of the perturbed problem is obtained in terms of noninteger powers of the perturbation parameter. Marked sensitivity of the eigensolutions is highlighted. Particular attention is devoted to the eigenvectors of the perturbed system and to the strong coupling that occurs between the chains. An example is developed to illustrate the algorithm and compare perturbative and numerical solutions.

I. Introduction

BECAUSE of the increasing importance of sensitivity analysis in structural dynamics, many methods have been developed to compute eigensolution derivatives with respect to system parameters. Weissenburger¹ and then Pomazal and Snyder² have studied the problem concerning a general system, but they have solved (in closed form) only the problem of local modification, consisting of a perturbation of a unique coefficient of the dynamic matrix. More general modifications have also been studied, even for nonsymmetric matrices in nonconservative problems. Perturbation methods or equivalent methods of explicit differentiation have been employed. References can be found in the review by Adelman and Haftka³ or in the recent book by Brandon.⁴ In the last few years, multiple eigenvalues have been studied, but attention has been focused on symmetric operators only (see the papers by Mills-Curran^{5,6} and Dailey⁷). The author is not aware of studies made on nonsymmetric matrices with repeated eigenvalues, which indeed present some specific problems because they usually exhibit an incomplete set of eigenvectors.

By applying results known in the algebraic functions theory, it is shown in Ref. 8 that the eigenvalues of the defective matrices depend on fractional powers of the perturbation parameter (see also Ref. 9); however, little attention is devoted to the associated eigenvectors, and no general computational method is given.

In this paper, after a brief review of the algebraic properties of defective matrices, a perturbative algorithm is developed for systematically computing their directional eigenderivatives. The method highlights the marked sensitivity of the eigensolutions and explains the mechanism for generating a complete system of eigenvectors from an incomplete system. Different cases of unique or several chains of generalized eigenvectors associated with a given unperturbed eigenvalue are considered, and the strong interactions between the chains are studied. As an example, a defective matrix is considered to test the algorithm through a comparison with a numerical solution.

II. Generalized Eigenvectors of Defective Matrices

Let us consider the eigenvalue problem

$$(A_0 - \lambda_0 I)u = 0 \quad (1)$$

where A_0 is a real matrix. By hypothesis A_0 is defective, i.e., there exists at least one eigenvalue λ_0 with geometric multiplicity n less than the algebraic multiplicity m . Let us denote with u_i ($i = 1, 2, \dots, n$) the eigenvectors associated with λ_0 . It is known from algebra¹⁰ that n chains of generalized eigenvectors can be associated with λ_0 . The i th chain is made of d_i vectors u_{ik} ($k = 1, 2, \dots, d_i$) (d_i being referred to as the length of the chain) that satisfy the recurrent relations

$$(A_0 - \lambda_0 I)u_{ik} = u_{i,k-1} \quad (i = 1, 2, \dots, n; k = 1, 2, \dots, d_i) \quad (2)$$

The eigenvectors u_{ik} therefore span an invariant subspace. Index k is called the order of the eigenvector; first-order eigenvectors are eigenvectors in the proper sense, $u_{i1} = u_i$. It is possible to show that the sum of the lengths d_i of the chains is equal to m and that the generalized eigenvectors form a base for the space. In this base A_0 assumes the Jordan canonical form, in which n blocks are associated with λ_0 , each of dimensions $d_i \times d_i$ ($i = 1, 2, \dots, n$).

Let us consider now the adjoint problem

$$(A_0 - \lambda_0 I)^H v = 0 \quad (3)$$

where the hermitian inner product has been assumed. Eigenvectors v_i ($i = 1, 2, \dots, n$) associated with λ_0 are the left eigenvectors of A_0 . For problem (3) it is possible also to find n chains of left generalized eigenvectors of the same lengths d_i ; it ensues that $v_{id_i} = v_i$.

If suitable normalization conditions are introduced, the following orthogonality conditions hold between left and right generalized eigenvectors:

$$v_{i1}^H u_{ik} = \delta_{ij} \delta_{kl} \quad (4)$$

where δ_{ij} is the Kronecker symbol. Equations (4) show that the first $d_i - 1$ right eigenvectors of the i th chain are orthogonal to all the proper left eigenvectors v_j ; so, in accordance with Eqs. (2), they are in the range of the operator, $u_{ik} \in \mathcal{R}(A_0 - \lambda_0 I)$ for $k < d_i$. On the contrary, all of the eigenvectors of maximum order are out of range, $u_{id_i} \notin \mathcal{R}(A_0 - \lambda_0 I)$. These properties will be used in the following discussion.

III. Eigensolutions of Nearly Defective Matrices

Let us study now the perturbed eigenvalue problem

$$(A - \lambda I)w = 0 \quad (5)$$

obtained by adding small perturbations to the matrix A_0 :

$$A = A_0 + \epsilon B \quad (6)$$

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In Eq. (6) $\|B\|$ and $\|A\|$ are of the same order and $0 < \epsilon \ll 1$; negative values of ϵ can be considered by substituting B by $-B$. Matrix A will be referred to as nearly defective; it generally has a complete system of eigenvectors, by excluding some special cases of perturbations. Its eigensolutions $\lambda(\epsilon)$ and $w(\epsilon)$ can be determined through a perturbation procedure by assuming they may be generated from the eigenvalues $\lambda_0 = \lambda(0)$ and the eigenvectors $u = w(0)$ of the non perturbed matrix A_0 . However, the main difficulty of the problem consists in formulating an algorithm that allows the generation of a complete system of eigenvectors from an incomplete system. This circumstance clearly shows that some nonlinear equations must appear at some level of the perturbation procedure. So a standard approach based on an expansion of the eigensolutions in terms of integer powers of ϵ , where all the equations are linear in the unknown coefficients of the series, necessarily fails.¹¹ Indeed, as discussed in Refs. 8 and 9, the eigensolutions of the perturbed problem must be expanded in series of fractional powers of the perturbative parameter, of type $\epsilon^{1/\nu}$, where ν is an integer which is not a priori known. Starting from this achieved result, a systematic method for computing the series terms is illustrated here.

Let us introduce a new perturbation parameter

$$\eta = \epsilon^{1/\nu} \quad (7)$$

and then expand $\lambda(\eta)$ and $w(\eta)$ in series of η around the point $\eta = 0$:

$$\lambda = \lambda_0 + \eta\lambda_1 + \eta^2\lambda_2 + \dots, \quad w = w_0 + \eta w_1 + \eta^2 w_2 + \dots \quad (8)$$

By substituting Eqs. (6-8) in Eq. (5) and equating to zero terms with the same power of η , the following perturbation equations are obtained:

$$\begin{aligned} \eta^0: & (A_0 - \lambda_0 I)w_0 = 0 \\ \eta^1: & (A_0 - \lambda_0 I)w_1 = \lambda_1 w_0 \\ \eta^2: & (A_0 - \lambda_0 I)w_2 = \lambda_1 w_1 + \lambda_2 w_0 \\ \eta^p: & (A_0 - \lambda_0 I)w_p = \lambda_1 w_{p-1} + \lambda_2 w_{p-2} + \dots + \lambda_p w_0 \\ \eta^\nu: & (A_0 - \lambda_0 I)w_\nu = \lambda_1 w_{\nu-1} + \lambda_2 w_{\nu-2} + \dots + \lambda_\nu w_0 - Bw_0 \\ \eta^{\nu+1}: & (A_0 - \lambda_0 I)w_{\nu+1} = \lambda_1 w_\nu + \lambda_2 w_{\nu-1} + \dots + \lambda_{\nu+1} w_0 - Bw_1 \end{aligned} \quad (9)$$

To avoid undeterminate quantities Eqs. (9) must be equipped with suitable normalization conditions. Here the condition $e_h^T w = 1$ is adopted, with e_h h th (properly selected) canonical vector; so $e_h^T w_0 = 1$, $e_h^T w_p = 0$ ($p > 1$) follow from Eq. (8b). Three different cases will be considered:

Case 1: there is a unique chain of generalized eigenvectors associated with λ_0 ($n = 1$, $d = m$);

Case 2: there are n chains associated with λ_0 , all of different length ($d_1 > d_2 > \dots > d_n$); and

Case 3: there are n chains associated with λ_0 , some of equal length ($d_1 \geq d_2 \geq \dots \geq d_n$).

The three cases are treated separately, the integer ν being determined in each instance.

A. Case 1

There is a unique chain of eigenvectors u_{ik} of length $d = m$; to simplify the notation, index 1 will be omitted ($u_{1k} \equiv u_k$). By successively solving Eqs. (9) and by taking into account Eq. (2), it follows that:

$$\begin{aligned} w_0 &= u_1 \\ w_1 &= \lambda_1 u_2 \\ w_2 &= \lambda_1^2 u_3 + \lambda_2 u_2 \\ w_p &= \lambda_1^p u_{p+1} + (p-1)\lambda_1^{p-2} \lambda_2 u_p + \dots + \lambda_p u_2 \quad (p \leq m-1) \end{aligned} \quad (10)$$

where coefficients $\lambda_1, \lambda_2, \dots, \lambda_p$ are still unknown. In Eqs. (10) generalized eigenvectors have been normalized according to $e_h^T w_1 = 1$, $e_h^T w_p = 0$ ($p > 1$); normalization conditions on w_p are thus satisfied.

Note that the first m perturbative equations ($p = 0, 1, \dots, m-1$) admit a solution (although it is not unique), because the known term belongs to the range of the singular operator; on the contrary, the equation of order η^m contains on the right side the generalized eigenvector of maximum order of the chain, u_m , which does not belong to the range. If the perturbation term Bw_0 appeared at a level lower than m (corresponding to a choice $\nu < m$), the relevant equation could not be solved, because in general $Bw_0 \notin \mathcal{R}(A_0 - \lambda_0 I)$. On the contrary, by selecting $\nu = m$, the perturbation term appears in the same equation in which u_m appears for the first time, being multiplied by λ_1^m . Then, by suitable choice of λ_1 , the component of the known term external to the range can be removed. For this equation the solvability condition is nonlinear, and yields m roots λ_1 in the complex plane. On the contrary, by going on to the higher order, linear conditions in $\lambda_2, \lambda_3, \dots$, are found (see Case 1 in the Appendix). So, the unique generating solution of the defective matrix generates m different solutions for the nearly defective matrix.

B. Case 2

There are n chains of eigenvectors of different length d_i , which are assumed to be in decreasing order. The generating solution can be taken to be a linear combination with arbitrary coefficients of eigenvectors u_{i1} . At subsequent steps, each eigenvector generates eigenvectors of higher order belonging to the same i th chain. After d_n steps, the perturbation equation contains on the right side the generalized eigenvector of highest order of the shortest chain, u_{nd_n} , and cannot be solved. By ordering at this level the term Bw_0 , i.e., by choosing $\nu = d_n$, and enforcing the right hand member (r.h.m.) to be orthogonal to v_n , an equation in $\lambda_1^{d_n}$ is obtained, from which d_n solutions are drawn. The constants of linear combination can be determined (to within one, available for the normalization) from the orthogonality condition of the r.h.m. to the remaining $n-1$ left eigenvectors v_1, v_2, \dots, v_{n-1} . Going on to higher orders it must be observed that, because n orthogonality conditions must be enforced at each step, it is always necessary to sum a suitable solution of the associated homogeneous problem to the particular solutions of the perturbation equations; this practice ensures the availability of a sufficient number of arbitrary constants.

The whole procedure can be repeated by assuming a generating solution made of the first $n-1$ eigenvectors u_i only, by assigning the zero value to a constant of the linear combination. In this way perturbation equations can be solved up to order η^{d_n-1} , before a term that is out of the range appears. By placing Bw_0 at this level, i.e., by choosing $\nu = d_{n-1}$, from the orthogonality condition to v_{n-1} an equation in $\lambda_1^{d_{n-1}}$ is obtained from which d_{n-1} solutions are found. There are now not enough constants available to remove the other components in the directions of $v_1, v_2, \dots, v_{n-2}, v_n$, having assumed one coefficient of the linear combination to be zero. The problem can be overcome by introducing the solution of the homogeneous problem αu_n at the step $d_{n-1} - d_n$. The constant α is therefore available at step d_{n-1} to remove the component of the known term parallel to v_n .

The perturbation process can be applied n times, in turn assuming ν equal to d_n, d_{n-1}, \dots, d_1 ; correspondingly $d_n + d_{n-1} + \dots + d_1 = m$ independent solutions are found. The perturbation method then generates m solutions starting from $n < m$ solutions.

Summarizing, referring to the r th phase ($r = 1, 2, \dots, n$) of the algorithm, it is necessary to proceed as follows:

1) Let us assume $\nu = d_r$ in Eq. (7).

2) Let us take the following generating solution, with arbitrary constants α_{j0}

$$w_0 = \sum_{j=1}^r \alpha_{j0} u_{j1} \quad (11)$$

3) At each step $p = 1, 2, \dots$, (i. e., at the order η^p) add to the particular solution obtained by using Eqs. (2) the linear combination with arbitrary coefficients α_{jp}

$$\sum_{j=1}^r \alpha_{jp} u_{j1} \quad (p = 1, 2, \dots)$$

4) Starting from step $p = d_r - d_i$ ($i = r + 1, r + 2, \dots, n$), at each step add to the particular solution terms

$$\alpha_{ik} u_{i1} \quad (i = r + 1, r + 2, \dots, n; k = 0, 1, \dots)$$

Note that points 2. through 4. can be thus summarized: at the step $p = 0, 1, 2, \dots$ add to the particular solution a linear combination of the generalized eigenvectors u_{i1} for all i such that $d_i \geq d_r - p$.

5) At step $p = d_r$: a) from the normalization condition $e_h^T w_0 = 1$ and the orthogonality conditions between the known term and the $r - 1$ left eigenvectors v_j ($j = 1, 2, \dots, r - 1$) the r constants α_{j0} are found; b) from the orthogonality condition to the left eigenvector v_r , d_r different values of λ_1 are obtained; and c) from the orthogonality conditions to the $n - r$ left eigenvectors v_i ($i = r + 1, \dots, n$) the $n - r$ constants α_{i0} are drawn, for each one of the d_r values of λ_1 .

6) At step d_{r+1} , for each of the d_r solutions so far obtained: a) from $e_h^T w_1 = 0$ the first group of orthogonality conditions r constants α_{j1} are calculated; b) from the condition of the second group a unique λ_2 is obtained; and c) from the third group of conditions the $n - r$ constants α_{i1} are evaluated.

7) In general, at step d_{r+p} the unknowns α_{jp} , λ_{p+1} , and α_{ip} are found.

It should be noted that, in some special cases of perturbation, the algorithm can fail (e.g., if some matrix is singular). These exceptional cases have not been studied. Explicit formulas for computing second-order approximations of the eigensolutions are given in case 2 in the Appendix.

C. Case 3

There are n chains of eigenvectors, some of equal length. The phase r of the algorithm of case 2 can still be applied if $d_{r-1} < d_r < d_{r+1}$. On the contrary, if there are s chains of equal length d_r , i.e., $d_{r-1} < d_r = d_{r+1} = \dots = d_{r+s-1} < d_{r+s}$, the procedure must be slightly modified as follows.

Let us pose $v = d_r$ and assume as generating solution a linear combination not only of the $r - 1$ eigenvectors associated with the chains longer than the r th, but also of the s eigenvectors associated with the chains of equal length

$$w_0 = \sum_{j=1}^{r+s-1} \alpha_{j0} u_{j1} \quad (12)$$

The same points 3. and 4. of the algorithm of case 2 must be followed. At step $p = d_r$ of the perturbation procedure (point 5.), instead of a unique equation in the unknown $\lambda_1^{d_r}$, an $s \times s$ eigenvalue problem in $\lambda_1^{d_r}$ and (α_{j0}) is obtained. By solving it, s eigenvalues (α_{j0}) are determined, d_r values of λ_1 being associated with each. At the next step (point 6.), for each one of the $s \times d_r$ solutions obtained, a unique λ_2 and a unique (α_{j1}) are found. Analogous results are obtained at higher orders (see case 3 in the Appendix).

IV. Discussion

The perturbation method developed in the previous section permits the highlighting of some interesting aspects of the eigensolutions behavior of defective matrices.

1) The eigensolutions are very sensitive to perturbations. This is because exponents smaller than unity appear in their asymptotic expansions in terms of the perturbation parameter.

2) If a unique chain of d eigenvectors is associated with the eigenvalue λ_0 , the perturbed problem admits d eigenvalues lying in a disk of center λ_0 and radius of order $\epsilon^{1/d}$. The associated eigenvectors are nearly parallel to the unique eigenvector of the defective matrix. To within terms of order ϵ , the eigenvectors still lie in the invariant subspace spanned by the chain of the generalized eigenvectors.

3) If two or more chains are associated with λ_0 , each of length d_i , the perturbed problem admits as many groups of eigenvalues as there are chains. In each group there are d_i eigenvalues lying in concentric circles of radius of order ϵ^{1/d_i} , the larger circle being associated with the longest chain. The eigenvectors are also divided into groups that follow this rule: the eigenvectors of the first group are nearly parallel to the proper eigenvector of the longest chain; the vectors of the second group are nearly parallel to a vector (depending on perturbations) that lies in the plane spanned by the proper eigenvectors of the two longest chains; the vectors of the third group are nearly parallel to a vector that lies in the space spanned by the proper eigenvectors of the three longest chains, and so on. Therefore, there exists a hierarchic order not only inside an individual chain but also among different chains that is strongly coupled. Unlike the previous case, the perturbed eigenvectors do not belong to the invariant subspaces spanned by the individual chains, but rather to the subspace that is the direct sum of the single subspaces, to within components of order ϵ .

4) When two or more chains have the same length, a further, strong coupling between these chains is highlighted. The proper eigenvectors of the chains of equal length combine themselves to furnish preferential directions, depending on

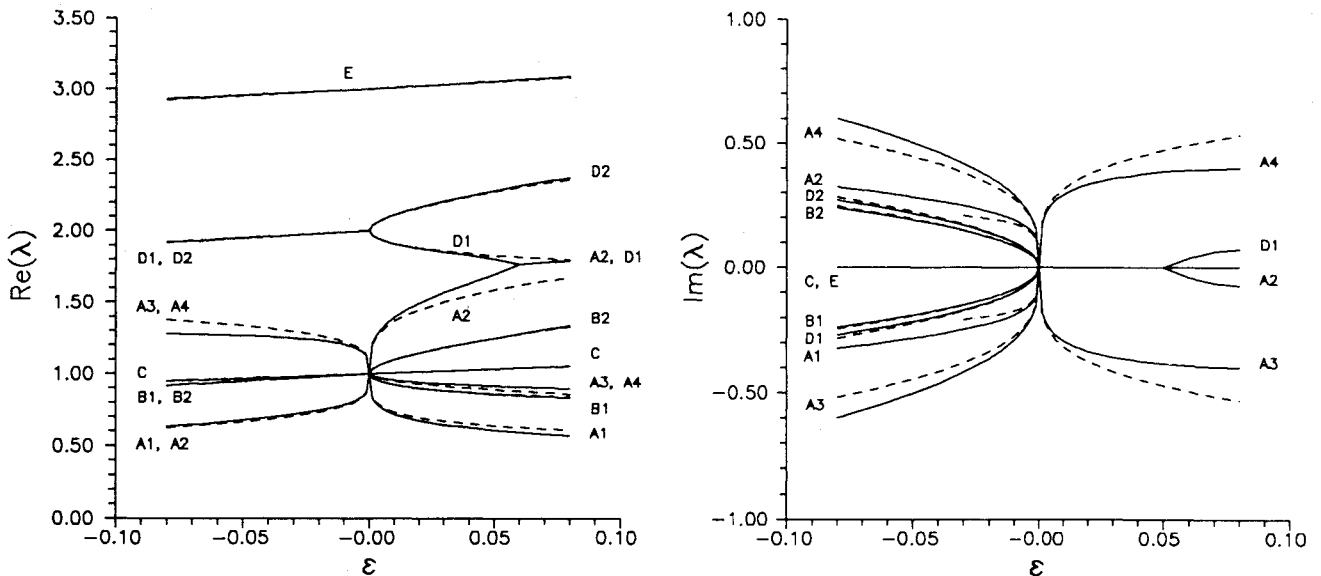


Fig. 1 Eigenvalues vs perturbation parameter (— numerical solution, --- perturbative solution).

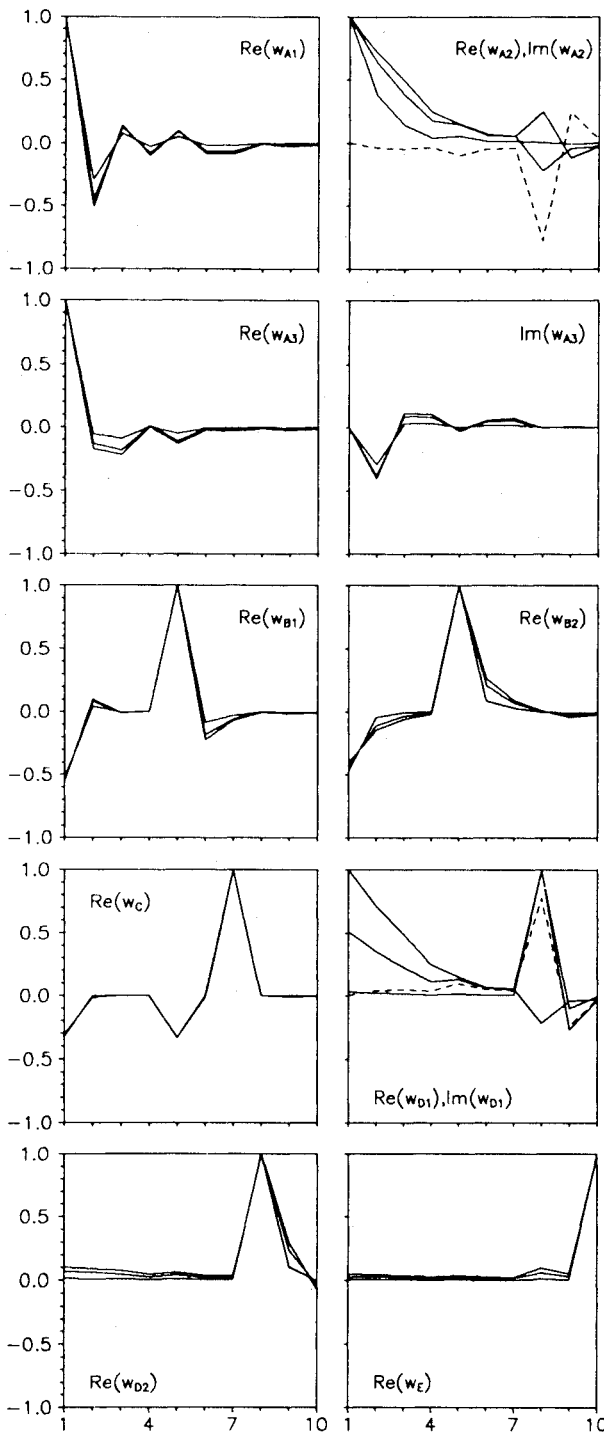


Fig. 2 Perturbed eigenvectors for $\epsilon = 0.01, 0.05$, and 0.08 (numerical solution).

perturbations, around which the perturbed eigenvectors are clustered.

5) The algorithm is unsuccessful if the perturbations have very special forms. In this case different forms of coupling could occur, but they will not be investigated here.

V. Illustrative Example

An example is developed to illustrate the algorithm and compare perturbative and numerical solutions. Let us consider a 10×10 matrix A_0 in the Jordan canonical form

$$A_0 = \text{diag} (J_A J_B J_C J_D J_E) \quad (13a)$$

$$J_A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$J_C = [1], \quad J_D = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J_E = [3] \quad (13b)$$

The matrix A_0 admits the real eigenvalues $\lambda_{01} = 1$, $\lambda_{02} = 2$, and $\lambda_{03} = 3$. The eigenvalue λ_{01} has multiplicities $m = 7$ and $n = 3$; the associated chains of right eigenvectors are $(e_1, e_2, e_3, e_4, (e_5, e_6), \text{ and } (e_7))$; and the left proper eigenvectors are e_4, e_6, e_7 . The eigenvalue λ_{02} has multiplicities $m = 2$ and $n = 1$, the unique right chain is (e_8, e_9) , and the left eigenvector e_{10} . Finally, λ_{03} is a simple eigenvalue with (right and left) eigenvector e_{10} . According to Eq. (6) let us perturb the matrix A_0 and solve the eigenvalue problem (5) by applying the perturbation method developed.

The eigenvalue λ_{01} will be considered first. The three associated chains have different lengths, so case 2 occurs. There are therefore three groups of eigenvalues (denoted by indices A, B, and C, respectively) that have the following expressions

$$\begin{aligned} \lambda_{Ak} &= 1 + \epsilon^{1/4} \lambda_{1k} + \epsilon^{1/2} \lambda_{2k} + \mathcal{O}(\epsilon^{3/4}) & (k = 1, \dots, 4) \\ \lambda_{Bk} &= 1 + \epsilon^{1/2} \lambda_{1k} + \epsilon \lambda_{2k} + \mathcal{O}(\epsilon^{3/2}) & (k = 1, 2) \\ \lambda_{Ck} &= 1 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \mathcal{O}(\epsilon^3) \end{aligned} \quad (14)$$

with associated nearly parallel eigenvectors

$$\begin{aligned} w_{Ak} &= e_1 + \mathcal{O}(\epsilon^{1/4}) & (k = 1, \dots, 4) \\ w_{Bk} &= \alpha_{10} e_1 + e_5 + \mathcal{O}(\epsilon^{1/2}) & (k = 1, 2) \\ w_{Ck} &= \alpha_{10} e_1 + \alpha_{20} e_5 + e_7 + \mathcal{O}(\epsilon) \end{aligned} \quad (15)$$

A unique chain is associated with λ_{02} (case 1) and therefore there is a single group of two solutions (of index D)

$$\begin{aligned} \lambda_{Dk} &= 2 + \epsilon^{1/2} \lambda_{1k} + \epsilon \lambda_{2k} + \mathcal{O}(\epsilon^{3/2}) & (k = 1, 2) \\ w_{Dk} &= e_8 + \mathcal{O}(\epsilon^{1/2}) & (k = 1, 2) \end{aligned} \quad (16)$$

A standard perturbation method can be applied to the simple eigenvalue λ_{03} to obtain the unique solution (of index E)¹¹

$$\begin{aligned} \lambda_E &= 3 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \mathcal{O}(\epsilon^3) \\ w_E &= e_{10} + \mathcal{O}(\epsilon) \end{aligned} \quad (17)$$

Expansions (14–17) have been obtained in Ref. 11 up to the second derivatives; details of the calculus will not be given here.

A specific application is considered, by fixing matrix B and varying the amplitude ϵ . Matrix B has been chosen in such a way that $|b_{ij}|_{\max} = 1$ [11]; the perturbation parameter has been varied in the interval $[-0.08, 0.08]$. Perturbation solutions and “exact” (numerical) solutions are compared.

The eigenvalues loci are shown in Fig. 1, where the real and imaginary parts of λ are plotted vs ϵ . The curves exhibit singular points at $\epsilon = 0$, that, at the same time, are bifurcation and limit points with respect to ϵ .

The eigenvalues of group A, associated with the longest chain, are those that vary most rapidly, due to the fact that they depend on $\epsilon^{1/4}$. When $0 < \epsilon < 0.05$, two eigenvalues are real and two complex conjugate; when $\epsilon < 0$, they are complex conjugate in twos. When the perturbation is of the order of 5%, the eigenvalues vary about 50%. The variations of the eigenvalues belonging to groups B and D are smaller, because they depend on $\epsilon^{1/2}$, but they are still marked. The modifications of the eigenvalues of groups C and E, associated with chains of unitary length, are very small, of the same order as the perturbation; the perturbed eigenvalues are still real, because λ_1 is determined from linear equations with real coefficients.

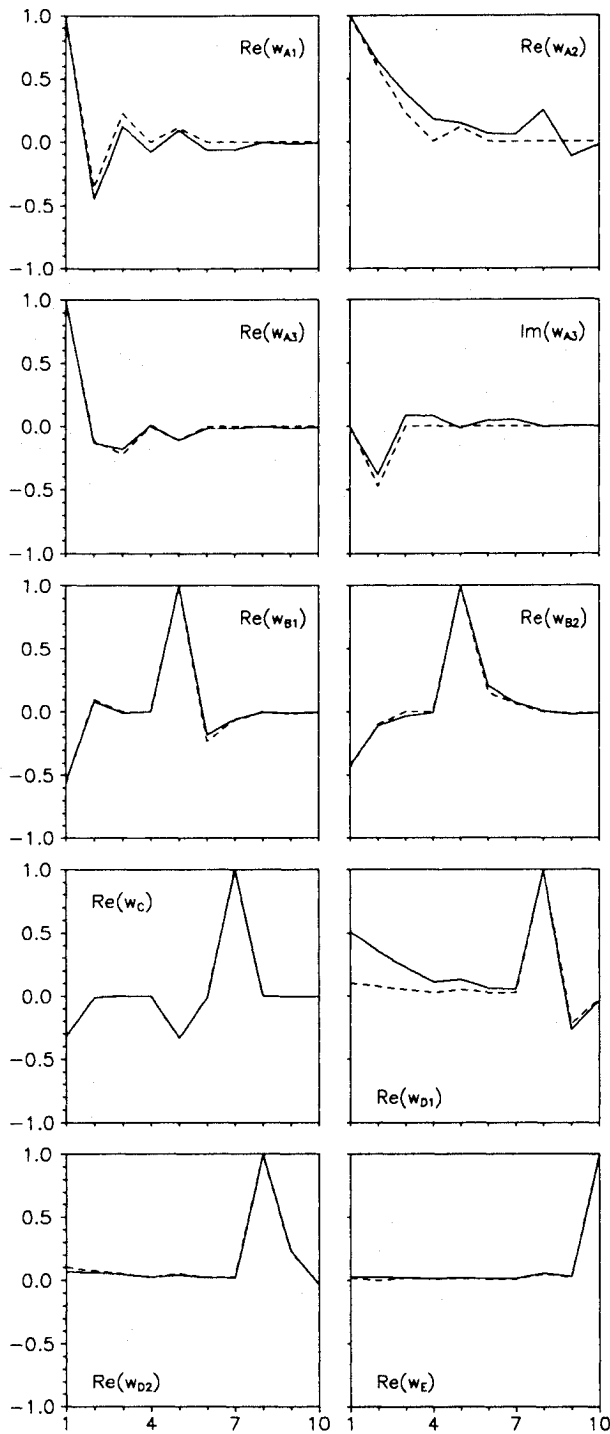


Fig. 3 Perturbed eigenvectors for $\epsilon = 0.05$ (— numerical solution, --- perturbative solution).

When $\epsilon = \bar{\epsilon} \approx 0.06$, the numerical solution shows a secondary bifurcation point between curves A2 and D1, associated with a multiple eigenvalue. This is because the eigenvalues A2 and D1, which are well separated in the nonperturbed problem, for increasing ϵ vary so rapidly that they become equal for a small $\epsilon = \bar{\epsilon}$, and then complex conjugate when $\epsilon > \bar{\epsilon}$. Obviously, this type of behavior cannot be described by the perturbation expansion, which is an extrapolation from the origin. However, apart from this aspect, the comparison of the two solutions is very satisfactory for all of the eigenvalues of groups B, C, D, and E, for some of which the error is not readable in the plot. On the contrary, differences are larger for the eigenvalues of group A, because as the perturbation ex-

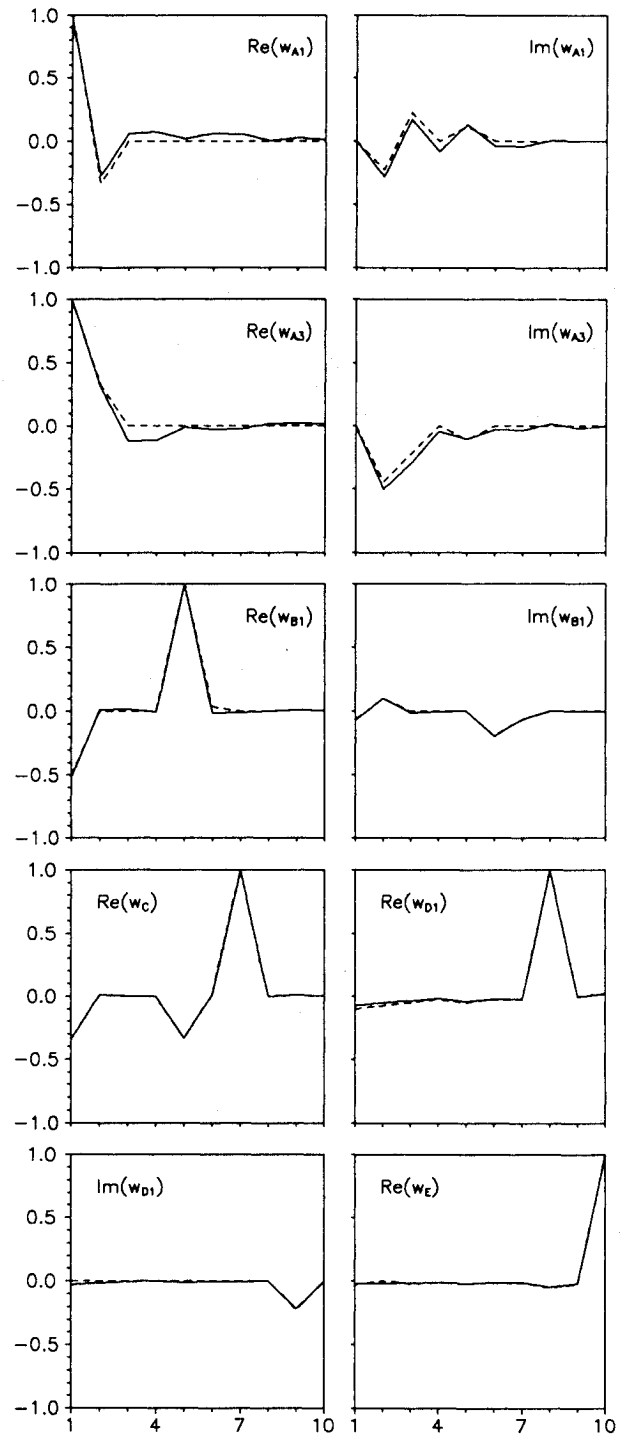


Fig. 4 Perturbed eigenvectors for $\epsilon = -0.05$ (— numerical solution, --- perturbative solution).

pansion is truncated at the order $\epsilon^{1/2}$, the error is of the order $\epsilon^{3/4}$, whereas in the other cases it is less than or equal to $\epsilon^{3/2}$.

Let us now pass on to examine the perturbed eigenvectors w . In Fig. 2 the exact eigenvectors corresponding to $\epsilon = 0.01$, 0.05, and 0.08 are shown. They are represented by joining with straight lines the points representing the ten components.

The four eigenvectors of group A are shown in the first two rows of Fig. 2. The eigenvector w_{A1} is always real; w_{A3} and w_{A4} are complex conjugate, so only the real and imaginary parts of w_{A3} are illustrated; w_{A2} is real for the two smallest values of ϵ and complex when $\epsilon = 0.08$; in this case its imaginary part is represented by a broken line. Curves show that the eigenvectors of this group have a prevailing component along e_1 (from

which they are generated in the perturbation process), and a component along e_2 (i.e., the second-order eigenvector of the chain) that increases rapidly with ϵ . In addition, coupling with e_8 is noticed, i.e., with the generator eigenvector of group D, in line with previous comments on Fig. 1.

The eigenvectors of group B have a prevailing component along vectors e_1 and e_5 , even when $\epsilon \rightarrow 0$, i.e., along the proper eigenvectors of the first two chains. In addition, an increasing contribution of second-order eigenvectors of the two chains e_2 and e_6 is noticed for increasing ϵ .

Eigenvector w_c is a linear combination of e_1 , e_5 , and e_7 , which are proper eigenvectors of the three chains associated with λ_{01} . Other vectors give vanishingly small contributions.

The eigenvectors of the group D, according to the theory developed, should have a prevailing component in the e_8 direction and a small correction along e_9 (second-order eigenvector) that increases with ϵ . This occurs only for w_{D2} , but not for w_{D1} , in which a strong coupling with e_1 is present and a decreasing contribution of e_2, e_3, \dots, e_7 manifests itself. This circumstance is a consequence of the occurrence of the secondary bifurcation point. When $\epsilon = 0.08$, $I_m(w_{D1})$ coincides, to within the sign, with $I_m(w_{A2})$.

Finally, eigenvector w_E is very slightly modified by perturbations, being nearly coincident with the eigenvector e_{10} .

In Fig. 3 the numerical solution relative to $\epsilon = 0.05$ and the perturbation solution are compared. Eigenvectors of group A are in good agreement; an excellent approximation is obtained for eigenvectors of groups B, C, and E and for w_{D2} . On the contrary, a marked error occurs in w_{D1} , because the perturbation solution cannot describe the interaction with the longest chain of eigenvectors.

The same analysis has been repeated for negative values of ϵ . All previous considerations hold; besides, depending on the fact that no secondary bifurcation points occur in the interval studied, no anomalous interactions occur other than those predicted by the perturbation analysis. This result is highlighted in Fig. 4 by the excellent accord between the two solutions for all eigenvectors of the groups B, C, D, and E, whereas the approximation for group A is still satisfactory.

VI. Conclusions

A sensitivity analysis of the eigensolutions of nonsymmetric matrices with multiple eigenvalues has been performed. A perturbation method based on series expansions of noninteger powers of the perturbation parameter has been illustrated. The method permits systematic derivation of the terms of the series up to the required order. In addition, it highlights the mechanism according to which an incomplete system of eigenvectors generates a complete system of nearly parallel eigenvectors. The following conclusions can be drawn:

1) The loci of the eigenvalues λ in terms of a perturbation parameter ϵ exhibit singular points at $\epsilon = 0$. They are bifurcation points, due to the multiplicity of the eigenvalues, and at the same time limit points with respect to ϵ , because $(d\lambda/d\epsilon)_{\epsilon=0} = \infty$. On the contrary, the derivatives of λ with respect to the new perturbation parameter $\eta = \epsilon^{1/\nu}$, with ν suitable selected integer, are finite.

2) As a consequence of the dependence of λ on $\epsilon^{1/\nu}$, eigenvalues of defective matrices are strongly sensitive to perturbations, because small changes in the coefficients of the matrix can produce very large variations in the eigenvalues. The sensitivity is closely connected to the length of the chains of the generalized eigenvectors associated with a multiple eigenvalue (i.e., with the dimensions of the Jordan blocks). The longer the chains, the more sensitive the eigenvalues.

3) When perturbations are introduced, the eigenvectors split and generate a complete system that spans the whole space. The perturbed eigenvectors are clustered around vectors that are linear combinations of the unperturbed eigenvectors. In particular, all the groups of eigenvectors have a first-order component in the direction of the proper eigenvector of the longest chain, whereas only one group has a first-order

component along the proper eigenvector of the shortest chain. Therefore the eigenvectors associated with a given multiple eigenvalue are organized in a hierarchical way, based on the length of the chain to which they belong.

4) The numerical example explained has shown that a two-term series expansion gives a good approximation to the exact results, if the lengths of the chains do not exceed a few units (e.g., five); otherwise, more terms are necessary, or a smaller region around the singular point $\epsilon = 0$ must be considered to limit the error. The perturbation expansion cannot, of course, take into account any eventual secondary bifurcation points that occur along the paths. In these cases the error can be larger.

5) The perturbation analysis of the eigensolutions can be performed by considering each eigenvalue separately. So, if one is interested in the perturbed eigensolutions close to a given unperturbed eigensolution, it is necessary only to know the chains associated with that eigenvalue. This analysis entails a small error of the order ϵ , with the exception of cases in which secondary bifurcations occur, which involve a stronger coupling between the chains.

6) It is expected that the motion of a dynamic system with defective matrix can be strongly affected by modifications of various kinds, including imperfections. Research is in progress on the subject.

Appendix: Eigenderivatives Expressions

Explicit formulas for evaluating first and second derivatives (with respect to $\eta = \epsilon^{1/\nu}$) of the eigensolutions of the perturbed problem (5) are given. The three cases discussed in Sec. III are considered separately.

A. Case 1

The perturbation equation of order η^m is

$$(A_0 - \lambda_0 I)w_m = \lambda_1^m u_m - Bu_1 + R \quad (A1)$$

where R stands for terms belonging to $\mathcal{R}(A_0 - \lambda_0 I)$. From the solvability condition it follows that

$$\lambda_1 = (v_m^H Bu_1)^{1/m} \quad (A2)$$

By solving Eq. (A1) we have

$$w_m = \hat{w}_m + (m-1)\lambda_1^{m-2}\lambda_2 u_m + \dots \quad (A3)$$

where \hat{w}_m is the solution corresponding to $R = 0$, satisfying $e_h^T \hat{w}_m = 0$. At the η^{m+1} order, the perturbation equation reads

$$(A_0 - \lambda_0 I)w_{m+1} = \lambda_1 \hat{w}_m + m\lambda_1^{m-1}\lambda_2 u_m - \lambda_1 Bu_2 + R \quad (A4)$$

from which emerges a linear condition on λ_2 :

$$\lambda_2 = v_m^H (B - \lambda_1 I) \hat{w}_1 / (m\lambda_1^{m-1}) \quad (A5)$$

B. Case 2

The perturbation equation of order η^{dr} is

$$(A_0 - \lambda_0 I)w_{dr} = \alpha_{r0}\lambda_1^{dr} u_{dr} + \sum_{i=r+1}^n \alpha_{i0}\lambda_1^{di} u_{di} - \sum_{j=1}^{r-1} \alpha_{j0} Bu_{j1} + R \quad (A6)$$

and the relevant solvability conditions are

$$-\sum_{j=1}^{r-1} \alpha_{j0} v_l^H Bu_{j1} = 0 \quad (l = 1, 2, \dots, r-1) \quad (A7a)$$

$$-\sum_{j=1}^{r-1} \alpha_{j0} v_r^H Bu_{j1} + \lambda_1^{dr} = 0 \quad (A7b)$$

$$-\sum_{j=1}^{r-1} \alpha_{j0} v_i^H Bu_{j1} + \alpha_{i0}\lambda_1^{di} = 0 \quad (i = r+1, r+2, \dots, n) \quad (A7c)$$

Provided $(c_{ij}) = (v_i^H B u_{j1})$ has rank $r - 1$, from Eqs. (A7) and the normalization condition $e_h^T w_0 = 1$, the unknowns α_{j0} , λ_1 , and α_{i0} can be obtained. Note that Eqs. (A7) are uncoupled.

By solving Eq. (A6) and denoting with \hat{w}_{d_r} the solution corresponding to $R = 0$, the successive perturbation equation is found to be

$$\begin{aligned} (A - \lambda_0 I) w_{d_r+1} &= \lambda_1 \hat{w}_{d_r} + \alpha_{r0} d_r \lambda_1^{d_r-1} \lambda_2 u_{rd_r} \\ &+ \lambda_1^{d_r} \alpha_{r1} u_{rd_r} + \delta_{r-1} \alpha_{r-1,0} \lambda_1^{d_r+1} u_{r-1,d_r-1} \\ &+ \sum_{i=r+1}^n (\alpha_{i0} d_i \lambda_1^{d_i-1} \lambda_2 + \alpha_{i1} \lambda_1^{d_i}) u_{id_i} - B w_1 + R \end{aligned} \quad (A8)$$

where (if $d_r > 1$)

$$w_1 = \sum_{j=1}^{r-1} (\alpha_{j0} \lambda_1 u_{j2} + \alpha_{j1} u_{j1}) + \delta_r \alpha_{r+1,0} u_{r+1,1} \quad (A9)$$

and

$$\delta_{r-1} = \begin{cases} 1 & \text{if } d_{r-1} - d_r = 1 \\ 0 & \text{if } d_{r-1} - d_r > 1 \end{cases} \quad (A10)$$

By imposing the solvability conditions on Eqs. (A8), the following equations are obtained:

$$\begin{aligned} & - \sum_{j=1}^{r-1} \alpha_{j1} v_i^H B u_{j1} - \lambda_1 v_i^H B \sum_{j=1}^{r-1} \alpha_{j0} u_{j2} \\ & - \delta_r \alpha_{r+1,0} v_i^H B u_{r+1,1} + \lambda_1 v_i^H \hat{w}_{d_r} \\ & + \delta_{i,r-1} \delta_{r-1} \alpha_{r-1,0} \lambda_1^{d_r+1} = 0 \quad (l = 1, 2, \dots, r-1) \end{aligned} \quad (A11a)$$

$$\begin{aligned} & - \sum_{j=1}^{r-1} \alpha_{j1} v_i^H B u_{j1} - \lambda_1 v_i^H B \sum_{j=1}^{r-1} \alpha_{j0} u_{j2} - \delta_r \alpha_{r+1,0} v_i^H B u_{r+1,1} \\ & + \lambda_1 v_i^H \hat{w}_{d_r} + \alpha_{r0} d_r \lambda_1^{d_r-1} \lambda_2 + \alpha_{r1} \lambda_1^{d_r} = 0 \end{aligned} \quad (A11b)$$

$$\begin{aligned} & - \sum_{j=1}^{r-1} \alpha_{j1} v_i^H B u_{j1} - \lambda_1 v_i^H B \sum_{j=1}^{r-1} \alpha_{j0} u_{j2} \\ & - \delta_r \alpha_{r+1,0} v_i^H B u_{r+1,1} + \lambda_1 v_i^H \hat{w}_{d_r} + \alpha_{i0} d_i \lambda_1^{d_i-1} \lambda_2 \\ & + \alpha_{i1} \lambda_1^{d_i} = 0 \quad (i = r+1, r+2, \dots, n) \end{aligned} \quad (A11c)$$

The α_{j1} , λ_2 , α_{i1} unknowns can be determined from Eq. (A11) and the normalization condition $e_h^T w_1 = 0$ for each solution of Eq. (A7). Note that the coefficients matrix of Eq. (A11a) is still (c_{ij}) , and Eqs. (A11) are uncoupled, like Eqs. (A7).

C. Case 3

Starting from the generating solution [Eq. (12)], the following perturbation equation of order η^{d_r} is found:

$$\begin{aligned} (A_0 - \lambda_0 I) w_{d_r} &= \lambda_1^{d_r} \sum_{j=r}^{r+s-1} \alpha_{j0} u_{jd_r} + \sum_{i=r+s}^n \alpha_{i0} \lambda_1^{d_i} u_{id_i} \\ &- \sum_{j=1}^{r+s-1} \alpha_{j0} B u_{j1} + R \end{aligned} \quad (A12)$$

It admits a solution if and only if the following relations are satisfied:

$$- \sum_{j=1}^{r+s-1} \alpha_{j0} v_i^H B u_{j1} = 0 \quad (l = 1, 2, \dots, r-1) \quad (A13a)$$

$$\begin{aligned} & - \sum_{j=1}^{r+s-1} \alpha_{j0} v_k^H B u_{j1} + \lambda_1^{d_r} \alpha_{k0} = 0 \\ & (k = r, r+1, \dots, r+s-1) \end{aligned} \quad (A13b)$$

$$\begin{aligned} & - \sum_{j=1}^{r+s-1} \alpha_{j0} v_i^H B u_{j1} + \alpha_{i0} \lambda_1^{d_i} = 0 \\ & (i = r+s, r+s-1, \dots, n) \end{aligned} \quad (A13c)$$

Equations (A13) are the counterpart of Eqs. (A7), valid in case 2. However, unlike those, the first two groups of equations are coupled. The first $r-1$ unknowns α_{j0} can be condensed by inverting a submatrix of (c_{ij}) previously defined, and an $s \times s$ eigenvalue problem can be obtained in the last s coefficients α_{j0} and the eigenvalue $\lambda_1^{d_r}$. By assuming that the s roots are distinct, s vectors α_{j0} (to within a constant) are found and $s \times d_r$ values for λ_1 . The s constants can be determined by enforcing $e_h^T w_0 = 1$ ($k = 1, 2, \dots, s$).

For each of the $s \times d_r$ solutions (α_{j0}, λ_1) it is possible to solve Eq. (A12) by finding a solution \hat{w}_{d_r} corresponding to $R = 0$. Then, the next perturbation equation reads

$$\begin{aligned} (A - \lambda_0 I) w_{d_r+1} &= \lambda_1 \hat{w}_{d_r} + d_r \lambda_1^{d_r-1} \lambda_2 \sum_{j=r}^{r+s-1} \alpha_{j0} u_{jd_r} \\ &+ \lambda_1^{d_r} \sum_{j=r}^{r+s-1} \alpha_{j1} u_{jd_r} + \delta_{r-1} \alpha_{r-1,0} \lambda_1^{d_r+1} u_{r-1,d_r-1} \\ &+ \sum_{i=r+s}^n (\alpha_{i0} d_i \lambda_1^{d_i-1} \lambda_2 + \alpha_{i1} \lambda_1^{d_i}) u_{id_i} - B w_1 + R \end{aligned} \quad (A14)$$

where δ_{r-1} is given by Eq. (A10) and (if $d_r > 1$)

$$w_1 = \sum_{j=1}^{r+s-1} (\alpha_{j0} \lambda_1 u_{j2} + \alpha_{j1} u_{j1}) + \delta_{r+s-1} \alpha_{r+s,0} u_{r+s,1} \quad (A15)$$

The following solvability conditions must hold:

$$\begin{aligned} & - \sum_{j=1}^{r+s-1} \alpha_{j1} v_i^H B u_{j1} - \lambda_1 \sum_{j=1}^{r+s-1} \alpha_{j0} v_i^H B u_{j2} \\ & - \delta_{r+s-1} \alpha_{r+s,0} v_i^H B u_{r+s,1} + \lambda_1 v_i^H \hat{w}_{d_r} \\ & + \delta_{i,r-1} \delta_{r-1} \alpha_{r-1,0} \lambda_1^{d_r+1} = 0 \quad (l = 1, 2, \dots, r-1) \end{aligned} \quad (A16a)$$

$$\begin{aligned} & - \sum_{j=1}^{r+s-1} \alpha_{j1} v_k^H B u_{j1} - \lambda_1 \sum_{j=1}^{r+s-1} \alpha_{j0} v_k^H B u_{j2} \\ & - \delta_{r+s-1} \alpha_{r+s,0} v_k^H B u_{r+s,1} + \lambda_1 v_k^H \hat{w}_{d_r} \\ & + d_r \lambda_1^{d_r-1} \lambda_2 \alpha_{k0} + \lambda_1^{d_r} \alpha_{k1} = 0 \\ & (k = r, r+1, \dots, r+s-1) \end{aligned} \quad (A16b)$$

$$\begin{aligned} & - \sum_{j=1}^{r+s-1} \alpha_{j1} v_i^H B u_{j1} - \lambda_1 \sum_{j=1}^{r+s-1} \alpha_{j0} v_i^H B u_{j2} \\ & - \delta_{r+s-1} \alpha_{r+s,0} v_i^H B u_{r+s,1} + \lambda_1 v_i^H \hat{w}_{d_r} \\ & + \alpha_{i0} d_i \lambda_1^{d_i-1} \lambda_2 + \alpha_{i1} \lambda_1^{d_i} = 0 \\ & (i = r+s, r+s+1, \dots, n) \end{aligned} \quad (A16c)$$

The first two groups of equations constitute a nonhomogeneous linear system of $r+s-1$ equations in the $r+s-1$ unknowns α_{j1} and in the parameter λ_1 . However, the matrix of coefficients is the same as that of the homogeneous equations (A13a, b), and therefore is singular. The system admits a solution if and only if the known term is orthogonal to the left eigenvector associated with λ_1 . From this condition, a single value of λ_2 is found for each of the $s \times d_r$ lower-order solutions previously obtained.

Acknowledgments

This research has been partially supported by The Italian Ministry of the University, Scientific and Technological Research 40 percent funds.

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